

BOREL SUBALGEBRAS OF THE WITT ALGEBRA W_1

YU-FENG YAO AND HAO CHANG

ABSTRACT. Let \mathbb{F} be an algebraically closed field of characteristic $p > 3$, and \mathfrak{g} the p -dimensional Witt algebra over \mathbb{F} . Let \mathcal{N} be the nilpotent cone of \mathfrak{g} . Explicit description of \mathcal{N} is given, so that the conjugacy classes of Borel subalgebras of \mathfrak{g} under the automorphism group are determined. In contrast with only one conjugacy class of Borel subalgebras in a classical simple Lie algebra, there are two conjugacy classes of Borel subalgebras in \mathfrak{g} . The representatives of conjugacy classes of Borel subalgebras, i.e., the so-called standard Borel subalgebras, are precisely given.

1. INTRODUCTION

As is well-known to all, Borel subalgebras play an important role in the structure and representation theory of a Lie algebra. Let \mathfrak{L} be a finite-dimensional simple Lie algebra over an algebraically closed field of characteristic 0. The famous structure theorem (cf. [5]) asserts that there is only one conjugacy class of Borel subalgebras under the automorphism group of \mathfrak{L} . The same result is also true for a classical simple Lie algebra over an algebraically closed field of prime characteristic with some mild restriction on the characteristic p . The classification theorem for finite-dimensional simple Lie algebras over an algebraically closed field of prime characteristic $p > 5$ implies that each finite-dimensional simple Lie algebra is either of classical type or of Cartan type (cf. [1, 10]). In this paper, we initiate the study of Borel subalgebras in the simple Lie algebras of Cartan type. We completely determine the conjugacy classes of Borel subalgebras in the Witt algebra, which is the “simplest” Lie algebra of Cartan type.

Let $\mathfrak{g} = W_1$ be the Witt algebra which was found by Witt as the first example of non-classical simple Lie algebra in 1930s. As is known to all, \mathfrak{g} is a restricted Lie algebra, and has a natural \mathbb{Z} -grading $\mathfrak{g} = \sum_{i=-1}^{p-2} \mathfrak{g}_{[i]}$. Associated with this grading, one has a filtration

2010 *Mathematics Subject Classification.* 17B05, 17B08, 17B50.

Key words and phrases. Witt algebra, Borel subalgebra, nilpotent element, nilpotent cone, automorphism group.

This work is supported by the National Natural Science Foundation of China (Grant Nos. 11201293 and 11271130), the Innovation Program of Shanghai Municipal Education Commission (Grant Nos. 13YZ077 and 12ZZ038), and the Fund of ECNU and SMU for Overseas Studies.

$(\mathfrak{g}_i)_{i \geq -1}$ with $\mathfrak{g}_i = \sum_{j \geq i} \mathfrak{g}_{[j]}$ for $i \geq -1$. Let $\mathcal{N} = \{x \in \mathfrak{g} \mid x^{[p]} = 0\}$ be the nilpotent cone of \mathfrak{g} , which is a closed subvariety in \mathfrak{g} . A. Premet extensively studied \mathcal{N} in [9], where he gave more general results on the nilpotent cone and the Chevalley Restriction Theorem for the Jacobson-Witt algebra. Quite recently, Premet's results were set up in the four classes of Lie algebras of Cartan type in [2]. Precise decomposition of \mathcal{N} into distinct nilpotent orbits under the automorphism group of \mathfrak{g} was given in [14]. In contrast with finitely many nilpotent orbits in a classical simple Lie algebra (cf. [6]), there are infinitely many nilpotent orbits in the Witt algebra. Moreover, M. Mygind [7] provided a complete picture of the orbit closures in the Witt algebra and its dual space, extending the results in [14].

This paper is structured as follows. After recalling some basic definitions and results on the Witt algebra, we explicitly describe nilpotent elements not contained in \mathfrak{g}_0 in section 3. We give a sufficient and necessary condition for a nilpotent element not contained in \mathfrak{g}_0 . This detailed description on nilpotent elements helps us to give a complete classification of conjugacy classes of Borel subalgebras of \mathfrak{g} in the final section. Our main result asserts that there are two conjugacy classes of Borel subalgebras in total. The representatives and dimensions are precisely given.

2. PRELIMINARIES

In this paper, we always assume that the ground field \mathbb{F} is algebraically closed, and of characteristic $p > 3$. Let $\mathfrak{A} = \mathbb{F}[x]/(x^p)$ be the truncated polynomial algebra of one indeterminate, where (x^p) denotes the ideal of $\mathbb{F}[x]$ generated by x^p . For brevity, we also denote by x the coset of x in \mathfrak{A} . There is a canonical basis $\{1, x, \dots, x^{p-1}\}$ in \mathfrak{A} . Let D be the linear operator on \mathfrak{A} subject to the rule $Dx^i = ix^{i-1}$ for $0 \leq i \leq p-1$. Denote by W_1 the derivation algebra of \mathfrak{A} . In the following, we always assume $\mathfrak{g} = W_1$ unless otherwise stated. By [12, §4.2], $\mathfrak{g} = \text{span}_{\mathbb{F}}\{x^i D \mid 0 \leq i \leq p-1\}$. There is a natural \mathbb{Z} -grading on \mathfrak{g} , i.e., $\mathfrak{g} = \sum_{i=-1}^{p-2} \mathfrak{g}_{[i]}$, where $\mathfrak{g}_{[i]} = \mathbb{F}x^{i+1}D$, $-1 \leq i \leq p-2$. Associated with this grading, one has the following natural filtration:

$$\mathfrak{g} = \mathfrak{g}_{-1} \supset \mathfrak{g}_0 \supset \dots \supset \mathfrak{g}_{p-2} \supset 0,$$

where

$$\mathfrak{g}_i = \sum_{j \geq i} \mathfrak{g}_{[j]}, \quad -1 \leq i \leq p-2.$$

This filtration is preserved under the action of the automorphism group G of \mathfrak{g} (cf. [3, 11, 13]). Furthermore, \mathfrak{g} is a restricted Lie algebra with the $[p]$ -mapping defined as the p -th power as

usual derivations. Precisely speaking,

$$(x^i D)^{[p]} = \begin{cases} 0, & \text{if } i \neq 1, \\ xD, & \text{if } i = 1. \end{cases}$$

We need the following result on the automorphism group of \mathfrak{g} .

Lemma 2.1. *(cf. [3, 13], see also [11, Theorem 12.8]) Let $\mathfrak{g} = W_1$ be the Witt algebra over \mathbb{F} and $G = \text{Aut}(\mathfrak{g})$. Then the following statements hold.*

- (i) *G is a connected algebraic group of dimension $p - 1$.*
- (ii) *$\text{Aut} \mathfrak{A} \cong G$, the correspondence is given by sending any $\phi \in \text{Aut} \mathfrak{A}$ to $\tilde{\phi} \in G$, where $\tilde{\phi}$ is defined via $\tilde{\phi}(\mathcal{D}) = \phi \circ \mathcal{D} \circ \phi^{-1}$, $\forall \mathcal{D} \in \mathfrak{g}$.*
- (iii) *G can be decomposed as $G = \mathbb{F}^\times \ltimes \mathbb{U}$, where \mathbb{F}^\times is the multiplicative group, and \mathbb{U} is the unipotent radical of G . More precisely, any element in G is of the form $\tilde{\varphi}$, where $\varphi \in \text{Aut} \mathfrak{A}$ is given as*

$$\varphi(x) = \sum_{i=1}^{p-1} a_i x^i, \quad a_i \in \mathbb{F}, \quad i = 1, \dots, p-1, \quad \text{and } a_1 \neq 0.$$

Moreover, $\tilde{\varphi} \in \mathbb{F}^\times$ if and only if $a_i = 0$ for $i = 2, \dots, p-1$. And $\tilde{\varphi} \in \mathbb{U}$ if and only if $a_1 = 1$.

Remark 2.2. Lemma 2.1 is not valid for $p = 3$. In fact, when $p = 3$, the Witt algebra $W_1 \cong \mathfrak{sl}_2$, and $\text{Aut}(\mathfrak{sl}_2)$ has dimension 3.

3. DESCRIPTION OF NILPOTENT ELEMENTS IN THE WITT ALGEBRA

Keep in mind that $\mathfrak{g} = W_1$ is the Witt algebra over \mathbb{F} . An element in \mathfrak{g} is called nilpotent if it is a nilpotent operator on \mathfrak{A} . Set $\mathcal{N} = \{x \in \mathfrak{g} \mid x^{[p]} = 0\}$. Then \mathcal{N} is just the set of all nilpotent elements in \mathfrak{g} . In the literature, \mathcal{N} is usually called the nilpotent cone, which is a closed subvariety in \mathfrak{g} . Then nilpotent cone \mathcal{N} was extensively studied by Premet in [9]. The following result is due to Premet.

Lemma 3.1. *(cf. [9, Theorem 2 and Lemma 4] or [14, Lemma 3.1]) Keep notations as above, then the following statements hold.*

- (i) *The orbit $G \cdot D$ is open and dense in \mathcal{N} . Moreover, it coincides with $(\mathfrak{g} \setminus \mathfrak{g}_0) \cap \mathcal{N}$.*
- (ii) *We have decomposition $\mathcal{N} = G \cdot D \cup \mathfrak{g}_1$.*

We are now in the position to give one of the main results describing nilpotent elements in the Witt algebra.

Proposition 3.2. *Let \mathfrak{g} be the Witt algebra with the nilpotent cone \mathcal{N} defined as above. Then*

$$D + \sum_{i=0}^{p-2} k_i x^{i+1} D \in \mathcal{N}$$

if and only if the following identity holds

$$(3.2.1) \quad k_{p-2} = \sum_{i=0}^{p-3} 2(i+1)k_i l_{p-2-i},$$

where l_i 's ($1 \leq i \leq p-2$) are defined inductively as follows.

$$(3.2.2) \quad l_1 = \frac{k_0}{2},$$

$$(3.2.3) \quad l_i = \frac{1}{i+1} \left(i k_{i-1} + \sum_{j=0}^{i-2} (2j+1-i) k_j l_{i-1-j} \right), \quad 2 \leq i \leq p-2.$$

Proof. (i) Suppose $X = D + \sum_{i=0}^{p-2} k_i x^{i+1} D \in \mathcal{N}$, then $X \in G \cdot D$ by Lemma 3.1(ii), i.e., there exists $\sigma \in G$ such that $X = \sigma(D)$. More precisely, $\sigma \in \mathbb{U}$, since $\sigma(D) - D \in \mathfrak{g}_0$. Let $Y = \sigma(xD)$. We can write down Y as follows

$$Y = xD + l_1 x^2 D + l_2 x^3 D + \cdots + l_{p-2} x^{p-1} D.$$

It is easy to check that

$$(3.2.4) \quad \begin{aligned} [X, Y] &= \left[D + \sum_{i=0}^{p-2} k_i x^{i+1} D, xD + \sum_{j=1}^{p-2} l_j x^{j+1} D_j \right] \\ &= D + 2l_1 xD + \sum_{t=1}^{p-3} \left((t+2)l_{t+1} + \sum_{s=0}^{t-1} (t-2s)k_s l_{t-s} - t k_t \right) x^{t+1} D \\ &\quad + \left(2k_{p-2} + \sum_{i=1}^{p-3} 2(i+1)k_{p-2-i} l_i + (p-2)k_0 l_{p-2} \right) x^{p-1} D. \end{aligned}$$

On the other hand,

$$(3.2.5) \quad [X, Y] = [\sigma(D), \sigma(xD)] = \sigma([D, xD]) = \sigma(D) = X = D + \sum_{i=0}^{p-2} k_i x^{i+1} D.$$

Comparing (3.2.4) with (3.2.5), we get the following relation

$$(3.2.6) \quad 2l_1 = k_0.$$

$$(3.2.7) \quad (t+2)l_{t+1} + \sum_{s=0}^{t-1} (t-2s)k_s l_{t-s} - tk_t = k_t, \quad 1 \leq t \leq p-3.$$

$$(3.2.8) \quad 2k_{p-2} + \sum_{s=1}^{p-3} 2(s+1)k_{p-2-s}l_s + (p-2)k_0l_{p-2} = k_{p-2}.$$

By (3.2.8), we have

$$\begin{aligned} k_{p-2} &= -(p-2)k_0l_{p-2} - \sum_{s=1}^{p-3} 2(s+1)k_{p-2-s}l_s \\ &= -(p-2)k_0l_{p-2} + \sum_{s=1}^{p-3} 2(p-1-s)k_{p-2-s}l_s \\ &= \sum_{i=0}^{p-3} 2(i+1)k_i l_{p-2-i} \end{aligned}$$

where l_i 's ($1 \leq i \leq p-2$) are defined by (3.2.6) and (3.2.7). It is easy to check that (3.2.6)-(3.2.7) are equivalent to (3.2.2)-(3.2.3).

(ii) Let

$$X = D + \sum_{i=1}^{p-2} k_i x^{i+1} D.$$

Suppose (3.2.1) holds. We need to show that $X \in \mathcal{N}$. For that, set

$$Y = xD + \sum_{i=1}^{p-2} l_i x^{i+1} D$$

with l_i 's defined by (3.2.2), (3.2.3). Following the same arguments as in part (i), it is a routine to check that $[X, Y] = X$. According to [4], there exists $\sigma \in \mathbb{U}$ such that $X' := \sigma(X) = D + cx^{p-1}D$ for some $c \in \mathbb{F}$. Let $Y' := \sigma(Y) = xD + \sum_{i=1}^{p-2} l'_i x^{i+1} D$. Then

$$[X', Y'] = [\sigma(X), \sigma(Y)] = \sigma([X, Y]) = \sigma(X) = X'.$$

By the same arguments as in part (i), we get a similar relation as (3.2.1) on the coefficients $\{0, 0, \dots, 0, c\}$ in the expression of X' as a linear span of $\{D, xD, \dots, x^{p-1}D\}$. This forces $c = 0$. Hence $\sigma(X) = D$, i.e., $X = \sigma^{-1}(D) \in \mathcal{N}$, as desired. \square

For any $(k_0, k_1, \dots, k_{p-3}) \in \mathbb{F}^{p-2}$, we define $(k'_1, \dots, k'_{p-2}) = (l_1, \dots, l_{p-2}) \in \mathbb{F}^{p-2}$ by (3.2.2)-(3.2.3). Thanks to Lemma 3.1 and Proposition 3.2, we get the following explicit description of nilpotent elements not contained in \mathfrak{g}_0 .

Theorem 3.3. *Let \mathfrak{g} be the Witt algebra, and G the automorphism group of \mathfrak{g} . Then*

$$G \cdot D = \{aD + \sum_{i=0}^{p-2} a^{-i} k_i x^{i+1} D \mid a \in \mathbb{F} \setminus \{0\}, (k_0, \dots, k_{p-3}) \in \mathbb{F}^{p-2}, k_{p-2} = \sum_{i=0}^{p-3} 2(i+1)k_i k'_{p-2-i}\}.$$

Proof. (i) Let $X \in G \cdot D$, then we can write

$$X = aD + \sum_{i=0}^{p-2} b_i x^{i+1} D, \quad a, b_i \in \mathbb{F}, i = 0, \dots, p-2, \text{ and } a \neq 0.$$

Take $\sigma \in G$ such that

$$\sigma(x^i D) = a^{i-1} x^i D, \quad 0 \leq i \leq p-1.$$

Then

$$(3.3.1) \quad \sigma(X) = D + \sum_{i=0}^{p-2} k_i x^{i+1} D,$$

where $k_i = a^i b_i$, $0 \leq i \leq p-2$. Since $\sigma(X) \in G \cdot D \subset \mathcal{N}$, we get by Proposition 3.2

$$k_{p-2} = \sum_{i=0}^{p-3} 2(i+1)k_i k'_{p-2-i}.$$

It follows from (3.3.1) that

$$\begin{aligned} X &= \sigma^{-1}\left(D + \sum_{i=0}^{p-2} k_i x^{i+1} D\right) \\ &= aD + \sum_{i=0}^{p-2} a^{-i} k_i x^{i+1} D. \end{aligned}$$

Hence,

$$G \cdot D \subseteq \{aD + \sum_{i=0}^{p-2} a^{-i} k_i x^{i+1} D \mid a \in \mathbb{F} \setminus \{0\}, (k_0, \dots, k_{p-3}) \in \mathbb{F}^{p-2}, k_{p-2} = \sum_{i=0}^{p-3} 2(i+1)k_i k'_{p-2-i}\}.$$

(ii) Let $X = aD + \sum_{i=0}^{p-2} a^{-i} k_i x^{i+1} D \in \mathfrak{g}$ with

$$a \in \mathbb{F} \setminus \{0\}, (k_0, \dots, k_{p-3}) \in \mathbb{F}^{p-2}$$

and

$$k_{p-2} = \sum_{i=0}^{p-3} 2(i+1)k_i k'_{p-2-i}.$$

Let $\sigma \in G$ such that

$$\sigma(x^i D) = a^{i-1} x^i D, \quad 0 \leq i \leq p-1.$$

Then

$$\sigma(X) = D + \sum_{i=0}^{p-2} k_i x^{i+1} D.$$

By Lemma 3.1 and Proposition 3.2, $\sigma(X) \in G \cdot D$, so that $X \in G \cdot D$. Hence,

$$\{aD + \sum_{i=0}^{p-2} a^{-i} k_i x^{i+1} D \mid a \in \mathbb{F} \setminus \{0\}, (k_0, \dots, k_{p-3}) \in \mathbb{F}^{p-2}, k_{p-2} = \sum_{i=0}^{p-3} 2(i+1)k_i k'_{p-2-i}\} \subseteq G \cdot D.$$

In conclusion, combining (i) with (ii), we get

$$G \cdot D = \{aD + \sum_{i=0}^{p-2} a^{-i} k_i x^{i+1} D \mid a \in \mathbb{F} \setminus \{0\}, (k_0, \dots, k_{p-3}) \in \mathbb{F}^{p-2}, k_{p-2} = \sum_{i=0}^{p-3} 2(i+1)k_i k'_{p-2-i}\},$$

as desired. \square

As a direct consequence, we have

Corollary 3.4. *Keep notations as before. Then the following statements hold.*

- (i) *The nilpotent orbit $G \cdot D$ has dimension $p - 1$.*
- (ii) *We have the following decomposition for the nilpotent cone.*

$$\begin{aligned} \mathcal{N} &= \{aD + \sum_{i=0}^{p-2} a^{-i} k_i x^{i+1} D \mid a \in \mathbb{F} \setminus \{0\}, (k_0, \dots, k_{p-3}) \in \mathbb{F}^{p-2}, \text{ and} \\ &\quad k_{p-2} = \sum_{i=0}^{p-3} 2(i+1)k_i k'_{p-2-i}\} \cup \mathfrak{g}_1. \end{aligned}$$

Remark 3.5. Corollary 3.4 (i) was obtained by Premet in [9].

Let

$$X = \sum_{i=-1}^{p-2} k_i x^{i+1} D \in \mathfrak{g}$$

with $k_i \in \mathbb{F}$, $-1 \leq i \leq p-2$. It is easy to see that the matrix of X relative to the canonical basis $\{1, x, \dots, x^{p-1}\}$ of \mathfrak{A} is

$$A = \begin{pmatrix} 0 & k_{-1} & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & k_0 & 2k_{-1} & \cdots & \cdots & \cdots & 0 \\ 0 & k_1 & 2k_0 & \cdots & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & k_{p-2} & 2k_{p-3} & \cdots & \cdots & \cdots & (p-1)k_0 \end{pmatrix}$$

So, the corresponding characteristic polynomial of A is

$$\begin{aligned}
 (\clubsuit) \quad |\lambda \mathbf{id} - A| &= \begin{vmatrix} \lambda & -k_{-1} & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \lambda - k_0 & -2k_{-1} & \cdots & \cdots & \cdots & 0 \\ 0 & -k_1 & \lambda - 2k_0 & \cdots & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & -k_{p-2} & -2k_{p-3} & \cdots & \cdots & \cdots & \lambda - (p-1)k_0 \end{vmatrix} \\
 &= \lambda^p + f(k_{-1}, k_0, \cdots, k_{p-2})\lambda
 \end{aligned}$$

where the expression of the right hand side of (\clubsuit) follows from [9, Corollary 1], \mathbf{id} represents the identity transformation on \mathfrak{A} , and

$$f(k_{-1}, k_0, \cdots, k_{p-2}) = \begin{vmatrix} k_0 & 2k_{-1} & 0 & \cdots & \cdots & \cdots & 0 \\ k_1 & 2k_0 & 3k_{-1} & \cdots & \cdots & \cdots & 0 \\ k_2 & 2k_1 & 3k_0 & \cdots & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ k_{p-2} & 2k_{p-3} & 3k_{p-4} & \cdots & \cdots & \cdots & (p-1)k_0 \end{vmatrix}.$$

Hence,

$$X = \sum_{i=-1}^{p-2} k_i x^{i+1} D \in \mathcal{N}$$

if and only if $f(k_{-1}, k_0, \cdots, k_{p-2}) = 0$.

According to [8, Theorem 2] and the proof of [2, Corollary 6.10], there exists a homogeneous polynomial $\psi_0 \in \mathbb{F}[k_{-1}, k_0, \cdots, k_{p-2}]$ with $\deg \psi_0 = p-1$ such that $X^{[p]} = \psi_0 X$ for any $X = \sum_{i=-1}^{p-2} k_i x^{i+1} D \in \mathfrak{g}$. Hence, $X = \sum_{i=-1}^{p-2} k_i x^{i+1} D \in \mathcal{N}$ if and only if $\psi_0 = 0$. Moreover, if $X \notin \mathcal{N}$, then $g(\lambda) := \lambda^p - \psi_0 \lambda$ is a minimal polynomial of $X = \sum_{i=-1}^{p-2} k_i x^{i+1} D$ as a transformation on \mathfrak{A} . Since (\clubsuit) is a characteristic polynomial of X , it follows that $g(\lambda) \mid (\lambda^p + f(k_{-1}, k_0, \cdots, k_{p-2})\lambda)$. Thus, $g(\lambda) = \lambda^p + f(k_{-1}, k_0, \cdots, k_{p-2})\lambda$, i.e., $\psi_0 = -f(k_{-1}, k_0, \cdots, k_{p-2})$. In conclusion, $X^{[p]} = -f(k_{-1}, k_0, \cdots, k_{p-2})X$ for any $X = \sum_{i=-1}^{p-2} k_i x^{i+1} D \in \mathfrak{g}$.

As a direct consequence of Corollary 3.4, we have

Corollary 3.6. *Keep notations as above, then*

$$\begin{vmatrix} k_0 & 2k_{-1} & 0 & \cdots & \cdots & \cdots & 0 \\ k_1 & 2k_0 & 3k_{-1} & \cdots & \cdots & \cdots & 0 \\ k_2 & 2k_1 & 3k_0 & \cdots & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ k_{p-2} & 2k_{p-3} & 3k_{p-4} & \cdots & \cdots & \cdots & (p-1)k_0 \end{vmatrix} = 0$$

if and only if one of the following two cases occurs

- (i) $k_{-1} = k_0 = 0$.
- (ii) $k_{-1} \neq 0$ and there exists some $(\kappa_0, \dots, \kappa_{p-3}) \in \mathbb{F}^{p-2}$ such that $k_i = k_{-1}^{-i} \kappa_i$ for $0 \leq i \leq p-3$ and $k_{p-2} = k_{-1}^{2-p} \sum_{i=0}^{p-3} 2(i+1) \kappa_i \kappa'_{p-2-i}$.

Proof. Since $X^{[p]} = -f(k_{-1}, k_0, \dots, k_{p-2})X$ for any $X = \sum_{i=-1}^{p-2} k_i x^{i+1} D \in \mathfrak{g}$, it follows that

$$f(k_{-1}, k_0, \dots, k_{p-2}) = \begin{vmatrix} k_0 & 2k_{-1} & 0 & \cdots & \cdots & \cdots & 0 \\ k_1 & 2k_0 & 3k_{-1} & \cdots & \cdots & \cdots & 0 \\ k_2 & 2k_1 & 3k_0 & \cdots & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ k_{p-2} & 2k_{p-3} & 3k_{p-4} & \cdots & \cdots & \cdots & (p-1)k_0 \end{vmatrix} = 0$$

if and only if

$$X = \sum_{i=-1}^{p-2} k_i x^{i+1} D \in \mathcal{N}.$$

Now the assertion follows directly from Corollary 3.4. \square

4. BOREL SUBALGEBRAS OF THE WITT ALGEBRA

As is well-known to all, Borel subalgebras play a fundamental role in the structure and representation theory of a Lie algebra. In this section, we determine the conjugacy classes of Borel subalgebras in the Witt algebra \mathfrak{g} over an algebraically closed field of prime characteristic $p > 3$.

Definition 4.1. A Borel subalgebra of a Lie algebra is defined to be a maximal solvable subalgebra.

Recall that \mathfrak{g} has a \mathbb{Z} -grading $\mathfrak{g} = \sum_{i=-1}^{p-2} \mathfrak{g}_{[i]}$. Set

$$\mathcal{B}^+ = \mathfrak{g}_0 = \sum_{i=0}^{p-2} \mathfrak{g}_{[i]} = \sum_{i=0}^{p-2} \mathbb{F}x^{i+1}D,$$

and

$$\mathcal{B}^- = \sum_{i=-1}^0 \mathfrak{g}_{[i]} = \mathbb{F}D \oplus \mathbb{F}xD.$$

It is easy to check that \mathcal{B}^+ and \mathcal{B}^- are Borel subalgebras. Moreover, they are not conjugate to each other. We call \mathcal{B}^+ and \mathcal{B}^- standard Borel subalgebras. The following result asserts that there are only two conjugacy classes of Borel subalgebras under the automorphism group G of \mathfrak{g} . They are represented by \mathcal{B}^+ and \mathcal{B}^- .

Theorem 4.2. *Let $\mathfrak{g} = W_1$ be the Witt algebra over an algebraically closed field \mathbb{F} of characteristic $p > 3$. Then any Borel subalgebra in \mathfrak{g} is conjugate to a standard Borel subalgebra under the action of the automorphism group of \mathfrak{g} .*

Proof. Let B be a Borel subalgebra of \mathfrak{g} . We divide the discussion into two cases.

Case 1: $B \subseteq \mathfrak{g}_0$.

In this case, $B = \mathcal{B}^+$, since $\mathfrak{g}_0 = \mathcal{B}^+$.

Case 2: $B \not\subseteq \mathfrak{g}_0$.

In this case, we claim that B is conjugate to \mathcal{B}^- . For that, we consider the intersection of B with the nilpotent cone \mathcal{N} .

(i) $B \cap \mathcal{N} \neq 0$.

In this situation, let $0 \neq u \in B \cap \mathcal{N}$, then $u \in G \cdot D$ or $u \in \mathfrak{g}_1$ by Lemma 3.1.

(i-1) Suppose $u \in G \cdot D$, then there exists $\sigma \in G$ such that $D \in \sigma(B)$. This implies that $\sigma(B) = \mathcal{B}^-$. Indeed, if there exists $v \in \sigma(B) \setminus \mathcal{B}^-$, we can write

$$v = \sum_{i=-1}^{p-2} a_i x^{i+1} D$$

with $a_i \neq 0$ for some $i > 0$. Set

$$j = \max \{l > 0 \mid a_l \neq 0\}.$$

Then

$$(\text{ad } D)^{j-1}(v) = \frac{(j+1)!}{2} a_j x^2 D + j! a_{j-1} x D + (j-1)! a_{j-2} D \in \sigma(B),$$

$$(\text{ad } D)^j(v) = (j+1)! a_j x D + j! a_{j-1} D \in \sigma(B).$$

Therefore, $\sigma(B)$ contains a semisimple subalgebra $\text{span}_{\mathbb{F}}\{D, xD, x^2D\} \cong \mathfrak{sl}_2$. It contradicts with the solvability of the subalgebra $\sigma(B)$.

(i-2) Suppose $u \in \mathfrak{g}_1$, then we will get a contradiction. Since $B \not\subseteq \mathfrak{g}_0$, there exists $u' \in B \setminus \mathfrak{g}_0$. According to [4], we can find $\sigma \in G$ such that $\sigma(u') = D + cx^{p-1}D$ for some $c \in \mathbb{F}$. Write

$$\sigma(u) = \sum_{j=i}^{p-2} k_j x^{j+1} D \text{ for some } i \geq 1, \text{ and } k_i \neq 0.$$

Then

$$(\text{ad}(\sigma(u')))^{i-1}(\sigma(u)) = \frac{(i+1)!}{2} k_i x^2 D + w_2 \in \sigma(B),$$

and

$$(\text{ad}(\sigma(u')))^i(\sigma(u)) = (i+1)! k_i x D + w_1 \in \sigma(B),$$

where $w_1 \in \mathfrak{g}_1$, $w_2 \in \mathfrak{g}_2$. Hence, $\sigma(B)$ contains a subalgebra

$$\text{span}_{\mathbb{F}}\{D + cx^{p-1}D, (i+1)! k_i x D + w_1, \frac{(i+1)!}{2} k_i x^2 D + w_2\}$$

which is not solvable by a direct check. This contradicts with the solvability of the subalgebra $\sigma(B)$.

(ii) $B \cap \mathcal{N} = 0$.

In this situation, $B \cap \mathfrak{g}_1 = 0$, so that $\dim B \leq \dim \mathfrak{g} - \dim \mathfrak{g}_1 = 2$.

(ii-1) If $\dim B = 1$. We can write $B = \text{span}_{\mathbb{F}}\{X\}$, where $X \in \mathfrak{g} \setminus \mathfrak{g}_0$. According to [8, Theorem 2], $X^{[p]} = cX$ for some $c \in \mathbb{F}$. Since $B \cap \mathcal{N} = 0$, we get $c \neq 0$, i.e., X is a semisimple element. It follows that \mathfrak{g} can be decomposed as a direct sum of eigensubspaces of $\text{ad } X$. Take any eigenvector $Y \neq X$, then $\text{span}_{\mathbb{F}}\{X, Y\}$ is a two-dimensional solvable subalgebra containing B . It contradicts with the maximality of B as a solvable subalgebra.

(ii-2) If $\dim B = 2$. By the structure of two-dimensional Lie algebras and the assumption above, we can choose a basis $\{X, Y\}$ of B with

$$X = D + \sum_{i=0}^{p-2} k_i x^{i+1} D, \quad Y = xD + \sum_{i=1}^{p-2} l_i x^{i+1} D, \quad \text{and } [X, Y] = X.$$

A similar argument as part (i) in the proof of Proposition 3.2 yields $X \in \mathcal{N}$. It contradicts with the assumption $B \cap \mathcal{N} = 0$.

In conclusion, B is conjugate to \mathcal{B}^+ or \mathcal{B}^- . More precisely, If $B \subseteq \mathfrak{g}_0$, then B is conjugate to \mathcal{B}^+ . While if $B \not\subseteq \mathfrak{g}_0$, then B is conjugate to \mathcal{B}^- . This completes the proof. \square

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DEPARTMENT OF MATHEMATICS, SHANGHAI MARITIME UNIVERSITY, SHANGHAI, 201306, CHINA.
E-mail address: yfyao@shmtu.edu.cn

DEPARTMENT OF MATHEMATICS, EAST CHINA NORMAL UNIVERSITY, SHANGHAI, 200241, CHINA.
E-mail address: hchang@ecnu.cn